



TITLE:

Unstable Cohomology Operations (Topics in Homotopy Theory and Cohomology Theory)

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Unstable Cohomology Operations

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Let $E^*(-)$ be a multiplicative generalized cohomology theory. This is represented by a spectrum E which can be represented as an Ω -spectrum

$$E_* = \{ \underline{E}_k \}_k, \quad \Omega \underline{E}_{k+1} \simeq \underline{E}_k.$$

Then we have

$$E^k X \simeq [X, \underline{E}_k],$$

or

$$E^* X = [X, \underline{E}_*].$$

We are interested in the unstable $E^*(-)$ cohomology operations, or the natural transformations

$$E^k X \longrightarrow E^n X.$$

We have that

$$\begin{array}{ccc} E^k X & \xrightarrow{\text{n.t.}} & E^n X \\ \simeq \updownarrow & & \updownarrow \simeq \\ [X, \underline{E}_k] & \xrightarrow{\text{n.t.}} & [X, \underline{E}_n], \end{array}$$

and so the natural transformations are given by

$$[\underline{E}_k, \underline{E}_n] \simeq E^n \underline{E}_k.$$

Consequently, $E^* \underline{E}_*$ is of interest. However, we will restrict our attention to additive operations, i.e. those r where

$$r(x+y) = r(x) + r(y).$$

To do this we will assume that

$$E^*(E_k \times E_k) \simeq E^*E_k \hat{\otimes}_{E^*E_k} E^*E_k.$$

Then the additive operations are just the primitives:

$$r \in PE^*E_* \quad \text{if} \quad r \mapsto r \hat{\otimes} 1 + 1 \hat{\otimes} r.$$

We can rigorously make PE^*E_* into a ring such that for any space X , E^*X is an "unstable E^*E module" over the ring PE^*E_* . The details will appear elsewhere but the concept is fairly clear. In the case of E^*X we have a map

$$PE^n E_k \otimes E^k X \longrightarrow E^n X$$

with a number of obvious compatibility conditions; among them the commuting of the diagram:

$$\begin{array}{ccc} PE^i E_n \otimes PE^n E_k \otimes E^k X & \longrightarrow & PE^i E_n \otimes E^n X \\ \downarrow & & \downarrow \\ PE^i E_k \otimes E^k X & \longrightarrow & E^i X, \end{array}$$

where the "ring" structure on PE^*E_* is clearly going to be given by composition of maps:

$$\begin{array}{ccc} PE^i E_n \otimes PE^n E_k & \longrightarrow & PE^i E_k \\ \cap & & \cap \\ [E_n, E_i] \otimes [E_k, E_n] & \longleftrightarrow & [E_k, E_i]. \end{array}$$

There is a map, cohomology suspension;

$$E^{k-n} E \longrightarrow PE^k E_n$$

from the stable operations to the unstable operations. This is just given by restricting a stable operation to classes of degree n .

An example of the potential usefulness is the nondesuspension problem.

If X has a desuspension $\Sigma^{-1}X$, then by the suspension isomorphism and the fact that stable operations commute with suspension, we have a stable E^*E module structure on $\tilde{E}^*(\Sigma^{-1}X)$. However, if $\Sigma^{-1}X$ exists, it must also have an unstable module structure compatible with the stable structure, i.e. we must be able to complete the diagram:

$$\begin{array}{ccc} E^{k-n}E \otimes \tilde{E}^n \Sigma^{-1}X & \longrightarrow & \tilde{E}^{k-n-1}X \\ \downarrow & \nearrow & \\ PE^k \underline{E}_n \otimes \tilde{E}^n \Sigma^{-1}X & & \end{array}$$

If this cannot be done, then $\Sigma^{-1}X$ does not exist.

We have specific examples for E in mind. In particular we want E to give complex cobordism or Brown-Peterson cohomology. The definition above, however, works for standard mod (p) cohomology as well.

In particularly nice cases,

$$E^*E_k \simeq \text{hom}_{E_*}(E_*E_k, E_*)$$

and

$$PE^*E_k \simeq \text{hom}_{E_*}(QE_*E_k, E_*).$$

Both BP and MU satisfy this property. Much more can be said. Hence forth,

let $E = \text{MU or BP}.$

In these cases

$$E_*(\underline{E}_k \times \underline{E}_k) \simeq E_*E_k \hat{\otimes}_{E_*} E_*E_k$$

and the diagonal map

$$\underline{E}_k \rightarrow \underline{E}_k \times \underline{E}_k$$

turns E_*E_k into a coalgebra.

Because \underline{E}_k is a homotopy commutative H-space, $E_*\underline{E}_k$ is a commutative Hopf algebra, with conjugation, over E_* ; or, in other words, an abelian group object in the category of coalgebras over E_* . Even more structure exists; since E_* is a ring spectrum we have maps

$$\underline{E}_k \wedge \underline{E}_n \longrightarrow \underline{E}_{k+n}$$

giving us a product

$$\circ : E_*\underline{E}_k \otimes_{E_*} E_*\underline{E}_n \longrightarrow E_*\underline{E}_{k+n},$$

and turning $E_*\underline{E}_* = \{E_*\underline{E}_k\}_k$ into a graded ring object over the category of coalgebras over E_* .

This goes as: E_*X is a graded ring, so \underline{E}_* is a graded ring object in the homotopy category, so $E_*\underline{E}_*$ is a graded ring object in the category of E_* -coalgebras.

The distributivity in this "ring", known as a "Hopf ring", uses the coproduct: let

$$x \longrightarrow \Sigma x' \times x'',$$

then

$$x \circ (y * z) = \Sigma \pm (x' \circ y) * (x'' \circ z)$$

where $*$ is the Hopf algebra product, or "addition" in our "ring".

$$E^*CP^\infty \simeq E^*[[x]] \quad \text{for } x \in E^2CP^\infty.$$

Dual to x^i we have $\beta_i \in E_2CP^\infty$.

We obtain a formal group law over E_* by applying $E^*(-)$ to the usual map

$$CP^\infty \times CP^\infty \rightarrow CP^\infty.$$

Then

$$x \rightarrow \sum_{i,j} a_{ij} x_1^i \otimes x_2^j = F(x_1, x_2).$$

Define

$$x +_F y = F(x, y) = \sum_{i,j} a_{ij} x^i y^j.$$

We define a few elements in E_*E_* .

Using

$$x \in E^2 CP^\infty = [CP^\infty, E_2]$$

we define

$$b_i \equiv x_*(\beta_i) \in E_{2i}E_2.$$

Also for

$$a \in E^k = [pt., E_k]$$

we have

$$[a] \equiv a_*(1) \in E_{0-k}E_*$$

we define

$$x +_{[F]} y = \sum_{i,j} [a_{ij}] \circ x^{\circ i} \circ y^{\circ j}.$$

In "The Hopf ring for complex cobordism", Journal of Pure and Applied Algebra, 1977, Ravenel and Wilson prove the following about MU and BP. Let $b(s) = \sum_{i \geq 0} b_i s^i$.

Theorem. In $E_*E_*[[s, t]]$, $E = MU$ or BP ,

$$b(s +_F t) = b(s) +_{[F]} b(t).$$

The Hopf ring E_*E_{2*} is generated over E_* by the b 's and $[E^*]$, and the only relations come from above. To obtain E_*E_* just add $e_1 \in E_1E_1$ and $e_1 \circ e_1 = b_1$. \square

These formulas, by duality, give all information about unstable MU and BP operations. However, there is another way to look at these unstable operations. For $n > 0$ we have the rational isomorphisms

$$E^*E_Q \simeq PE^*E_{-nQ}.$$

Since there is no torsion anywhere we have

$$\begin{array}{ccccc} E^{*+n}E \subset E^{*+n}E_Q & \simeq & \text{hom}_{E_*}(E_{*+n}E, E_{*Q}) \\ \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ PE^*E_{-n} \subset PE^*E_{-nQ} & \simeq & \text{hom}_{E_*}(QE_{*+n}, E_{*Q}) \end{array}$$

and we can represent an unstable operation by a rational stable operation, However, we have the following surprising result:

Theorem. For $E = MU$ or BP , the coker in

$$0 \rightarrow E^{*-n}E \rightarrow PE^*E_{-n} \rightarrow \text{coker} \rightarrow 0$$

has no torsion. \square

This may seem like a contradiction, but because of completion problems it is not. We have that

$$E^*E \simeq E^* \hat{\otimes} S$$

where S has only nonnegative degrees. E^* has only non positive degrees.

When we say "rationally" we mean

$$E^*E_Q \simeq E_Q^* \hat{\otimes} S,$$

not tensor product with Q . In this completed tensor product, an element which is non trivial in the coker is an infinite sum

$$\sum a_i \hat{\otimes} s_i, \quad a_i \in E_Q^*, \quad s_i \in S,$$

with the denominators of the a_i going to infinity as i does.

A candidate for an unstable operation can be checked now. If we are given an element of E_*E_Q we can evaluate it in

$$\text{hom}_{E_*}(E_*E_n, E_*Q)$$

and if we find that all of our values are really in

$$E_* \subset E_*Q,$$

then we have a legitimate element of

$$PE_*E_n.$$

It is at this stage that the detailed knowledge of E_*E_* developed in "The Hopf ring for complex cobordism" is useful.

An example of an unstable operation found in this way is the Adams operation ψ^k . These have been studied by several authors rationally, however we can obtain the following by use of the above technique.

Theorem. For $E = MU$ or BP , the rational operations $k^i \psi^k$ actually lie in

$$PE^{2i}_{E_{2i}} \text{ and } PE^{2i+1}_{E_{2i+1}}, \text{ all } i. \quad \square$$

In order to prove this type of result, techniques for evaluating

$$E_*(r) : E_*E_k \rightarrow E_*E_n \text{ for } r : E_k \rightarrow E_n$$

are necessary.

The details of these techniques, the last two theorems, and the rigorous definition of general unstable operations will appear elsewhere.

This paper represents a portion of the lectures I gave at a conference at the Research Institute for Mathematical Sciences at Kyoto University in October 1980. I would like to thank the participants and organizers for a most enjoyable conference.